Explicitly Valuing Mathematical Creativity in Proof-Based Courses

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MATHEMATICS

The Creativity Research Group

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The Scene in 2016

- Wolfram Alpha
- Web sites dedicated to textbook solutions
- Automatic theorem-provers
- Workforce change in the United States

The Scene in 2016 (cont.)

- Engineers are packaged as problem solvers rather than **creators and innovators** addressing grand challenges. (Sullivan, 2006)
- More than 1,500 Chief Executive Officers from 60 countries and 33 industries worldwide, believe that -- more than rigor, management discipline, integrity or even vision -- successfully navigating an increasing complex world will require **creativity**. (IBM 2010 Global CEO Study.)

Creativity in Guidelines & Standards

MAA CUPM 2015 Guidelines

- A successful major offers a program of courses to gradually and intentionally leads students from basic to advanced levels of critical and analytical thinking, ***while encouraging creativity*** and ***excitement about mathematics.***
- Major programs should include activities designed to promote students' progress in learning to approach mathematical problems with curiosity and ***creativity*** and ***persist*** in the face of difficulties.

Valuing Creativity

[I]n seeking to facilitate the development of talented young mathematicians, **neglecting** to recognize **creativity may drive** the creatively **talented underground or**, worse yet, cause them to **give up** the study of **mathematics** altogether. (Mann, 2005, p. 239).

Valuing Creativity

"It is in the best interest of the field of mathematics education that we identify and nurture creative talent in the mathematics classroom" (Sriraman, 2004, p. 32).

"For it is through mathematical creativity that we see the essence of what it means to 'do' and learn mathematics." (Liljedahl, 2009, p. 239).

Defining Creativity

What is mathematical creativity?

In particular, in proving?

Over 100 definitions of mathematical creativity! (Mann, 2006)

Defining Creativity

- Psycho-Analytic: Many mathematicians describe an **enlightenment** that is somewhat **unexpected** (Hadamard, 1945; Poincare, 1958).
- Product: Some focus on emphasizing whether the end-product is **original** and **useful** (Runco & Jaeger, 2012), perhaps to the mathematics field (Csikszentmihalyi, 1988).
- Process: describe it as a process that involves **different modes of thinking**, some of **an unusual** nature (Balka, 1974).

Process vs. Product

"It is important that when judging the creativity of a student we pay attention also to the ***process*** by which he[/she] arrived to the results and not only to the final problem." (Pelczer & Rodriguez, 2011, p. 394)

Absolute vs. Relative Creativity

- Absolute creativity versus Relative creativity
  - Historical inventions or discoveries at a global level
  - The discoveries by a specific person within a specific reference group, to human imagination that creates something new (Vygotsky, 1982, 1984)

K-12 Math Creativity

- Torrance (1966, 1978) created testing for creativity and giftedness in K-12 education
- Silver (1997) expanded three aspects of K-12 mathematical creativity
  - Flexibility - An ability to look at a problem from new perspective
  - Originality - Using an unexpected or unusual approach
  - Fluency - Applying ideas, tools of one area in a different area
- Elaboration – Expanding on each approach (Torrance, 1978)
- Iconoclasm – Affect, Self-efficacy of each approach (Chamberlain & Mann, 2015)

Situating Our Work

- Creativity in K-12 classrooms is different than the kind employed by mathematicians (Sriraman, 2005)
- Proof-writing aligns more with mathematicians' creativity

Overarching Research Question

How can we **explicitly value** and **foster potential for** undergraduate students' creativity in mathematics?

Specific Research Questions

- What properties/actions can undergraduate students learn/enact to generate potential for being creative in proving?
- How can we implement said properties/actions in the classroom?

Creativity-in-Progress Rubric (CPR) on Proving

- Creativity rubric from AAC&U (Rhodes, 2010)
- Leikin's (2009) Problem-Solving Rubric
- Interview with mathematicians (Tang, et al., 2015)
- Constant alpha-testing on students' LiveScribe work
- Feedback from past presentations
- Coding student process with mathematicians

AAC&U Rubric

CREATIVE THINKING VALUE RUBRIC

for more information, please contact value@aacu.org

Association of American Colleges and Universities

Definition

Creative thinking is both the capacity to combine or synthesize existing ideas, images, or expertise in original ways and the experience of thinking, reacting, and working in an imaginative way characterized by a high degree of innovation, divergent thinking, and risk taking.

Evaluators are encouraged to assign a zero to any work sample or collection of work that does not meet benchmark (cell one) level performance.

	Capstone 4	Milestones 3	 2	Benchmark 1
Acquiring Competencies *This step refers to acquiring strategies and skills within a particular domain.*	Reflect: Evaluates creative process and product using domain-appropriate criteria.	Create: Creates an entirely new object, solution or idea that is appropriate to the domain.	Adapt: Successfully adapts an appropriate exemplar to his/her own specifications.	Model: Successfully reproduces an appropriate exemplar.
Taking Risks *May include personal risk (fear of embarrassment or rejection) or risk of failure in successfully completing assignment, i.e. going beyond original parameters of assignment, introducing new materials and forms, tackling controversial topics, advocating unpopular ideas or solutions.*	Actively seeks out and follows through on untested and potentially risky directions or approaches to the assignment in the final product.	Incorporates new directions or approaches to the assignment in the final product.	Considers new directions or approaches without going beyond the guidelines of the assignment.	Stays strictly within the guidelines of the assignment.
Solving Problems	Not only develops a logical, consistent plan to solve problem, but recognizes consequences of solution and can articulate reason for choosing solution.	Having selected from among alternatives, develops a logical, consistent plan to solve the problem.	Considers and rejects less acceptable approaches to solving problem.	Only a single approach is considered and is used to solve the problem.
Embracing Contradictions	Integrates alternate, divergent, or contradictory perspectives or ideas fully.	Incorporates alternate, divergent, or contradictory perspectives or ideas in a exploratory way.	Includes (recognizes the value of) alternate, divergent, or contradictory perspectives or ideas in a small way.	Acknowledges (mentions in passing) alternate, divergent, or contradictory perspectives or ideas.
Innovative Thinking *Novelty or uniqueness (of idea, claim, question, form, etc.)*	Extends a novel or unique idea, question, format, or product to create new knowledge or knowledge that crosses boundaries.	Creates a novel or unique idea, question, format, or product.	Experiments with creating a novel or unique idea, question, format, or product.	Reformulates a collection of available ideas.
Connecting, Synthesizing, Transforming	Transforms ideas or solutions into entirely new forms.	Synthesizes ideas or solutions into a coherent whole.	Connects ideas or solutions in novel ways.	Recognizes existing connections among ideas or solutions.

Leikin (2009) Rubric

<table>
<tr><td colspan="2" rowspan="2"></td><td colspan="4">Creativity</td></tr>
<tr><td>Flu-ency</td><td>Flexibility</td><td>Originality</td><td></td></tr>
<tr><td rowspan="2">Scores per solution</td><td>For a problem independently of performance
or
For an individual student from a small group</td><td rowspan="2">1</td><td rowspan="2">$Flx_1 = 10$
for the first solution
$Flx_i = 10$
solutions from a different group of strategies
$Flx_i = 1$
similar strategy but a different representation
$Flx_i = 0.1$
the same strategy, the same representation</td><td>$Or_i = 10$
for insight/ unconventional solution
$Or_i = 1$
for model-based/ partly unconventional solution
$Or_i = 0.1$
for algorithm-based/ conventional solution</td><td rowspan="2">$Flx_i \times Or_i$</td></tr>
<tr><td>For student performance in a big group</td><td>$Or_i = 10 \quad P < 15\%$
$Or_i = 1 \quad 15\% \leq P < 40\%$
$Or_i = 0.1 \quad P \geq 40\%$</td></tr>
<tr><td colspan="2">Total score</td><td>n</td><td>$Flx = \sum_{i=1}^{n} Flx_i$</td><td>$Or = \sum_{i=1}^{n} Or_i$</td><td>$\sum_{i=1}^{n} Flx_i \times Or_i$</td></tr>
<tr><td colspan="2">Final creativity score</td><td colspan="4">$Cr = n\left(\sum_{i=1}^{n} Flx_i \times Or_i\right)$</td></tr>
</table>

n is the total number of appropriate solutions

$P = (m_j / n) \cdot 100\%$ where m_j *is the number of students who used strategy* j

CPR on Proving (cont.)

- ***Categories***
  (1) Making Connections
  (2) Taking Risks

- ***Levels (Continuum)***
  - Beginning
  - Developing
  - Advancing

Making Connections:

The ability to connect the proving task with definitions, theorems, multiple representations, and examples from the current course that a student is in, and possible prior experiences from previous courses.

Making Connections

- "[F]inally I found some nice books in an ***area totally unrelated to mine***, in matrix theory, and at some point I realized that ***I could apply this*** [aspect of Matrix Theory] ***that no one ever thought*** of applying to differential equations before and solved my problem … [I]n the process of applying it, I think I created … some new connections." - Dr. C

CPR on Proving

MAKING CONNECTIONS:	Beginning	Developing	Advancing
Between Definitions/Theorems	Recognizes some relevant definitions/theorems from the course or textbook with no attempts to connect them in their proving	Recognizes some relevant definitions/theorems from the course and attempts to connect them in their proving	Implements definitions/theorems from the course and/or other resources outside the course in their proving
Between Representations	Provides a representation with no attempts to connect it to another representation	Recognizes connections between representations	Uses connections between different representations
Between Examples	Generates one or two specific examples with no attempt to connect them	Recognizes a connection between the generated examples	Uses the key idea synthesized from generating examples

Taking Risks:

The ability to actively attempt a proof, demonstrate flexibility in using multiple approaches or techniques, posing questions about reasoning within the attempts, and evaluating those attempts.

Taking Risks

- "[O]ccasionally when you are trying to prove something, you know where you want to go, so it's just a matter of ***trying several different things***, and seeing what fits in order to get you there. But other times, you don't know where you are going. Proving means you're saying, "There is this problem, and ***I'm going to try this approach and this approach***. I don't even know what the next step should be." So I think the creativity part of it affects the proof differently." – Dr. B

CPR on Proving

TAKING RISKS:	Beginning	Developing	Advancing
Tools and Tricks[2]	Uses a tool or trick that is algorithmic or conventional for the course or the student	Uses a tool or trick that is model-based or partly unconventional[3] for the course or the student	Creates a tool or trick that is unconventional for the course or the student
Flexibility[4]	Begins a proof attempt (or more than one proof attempt), but uses only one approach	Acknowledges and/or uses more than one proving approach, but only draws on one proof technique	Uses more than one proof technique
Posing Questions	Recognizes there should be a question asked, but does not pose a question[5]	Poses questions clarifying a statement of a definition or theorem	Poses questions about reasoning within a proof
Evaluation of Proof Attempt	Examines surface-level[6] features of a proof attempt	Examines an entire proof attempt for logical or structural flow	Examines and *revises* an entire proof attempt for logical or structural flow

Correctness

"I will risk it and say that **[a proof] doesn't have to be correct to be creative**. But at least it [the proof] should be fixable. It can happen that you have an original idea and you mess up details, which is not surprising because if it is an original idea then it means that you haven't practiced that, [so] you would make mistakes."

What the CPR on Proving is and is not

- It is NOT assessing "correctness" or "validity" of the final proof.
- It is examining the full process of proof production.
- It is NOT a rubric to label student's creativity!
- It makes explicit some aspects that may promote mathematical creativity.

Example: Student 10

- Student 10 attempted to prove the statement, "If 3 divides the sum of the digits of n, then 3|n."
  - Third theorem in the number theory section of the course
    - Def S: $a|b \Leftrightarrow b = na$ for some $n \in Z$
    - (27) If m and n are even numbers, prove that $m + n$ and $m \cdot n$ are even numbers
    - (28) If $a|b$ and $a|c$, then $a|(br + cs)$ for any $r, s \in Z$.
- Three days of work provided by student 10 for the proof of this theorem.

Attempt 1

2a: Let $A = 10a + b$. Assume $3 \mid (A+B)$ specific (1)

Let $n = a_n 10^n + a_{n-1} 10^{n-1} + \dots + a_1 10^1 + a_0 10^0$ (2)

Induct? (3)

$123 = 100 + 20 + 3$ (4)

$= 1(99+1) + 2(9+1) + 3$ (5)

$= 99 + 1 + 18 + 2 + 3$ (6)

$= 99 + 18 + 6$ Use (28) (7)

Attempt 2: After Hint in Class

29: Let ~~n~~ $= a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_1 10^1 + a_0 10^0$ (8)

where $a_n, 10$ represent the digits of n. (9)

Suppose $3 \mid (a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_1 10^1 + a_0 10^0)$ (10)

$a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_1 10^1 + a_0 10^0$ (11)

$= a_n($ (12)

Attempt 3: Later That Night

29: ~~Claim~~ Thm: $3 \mid n$ if and only if 3 divides the addition of the digits of n. Let $n \in \mathbb{Z}$

n. Let $n = a_m 10^m + a_{m-1} 10^{m-1} + \dots + a_1 10^1 + a_0 10^0$

Let $s = a_m + a_{m-1} + \dots + a_1 + a_0$

Then $n - s = a_1 + \dots + a_{m-1}(10^{m-1} - 1^{m-1}) + 9^m a_m$

$= a_m 9^m + a_{m-1} 9^m + \dots + a_1 9$

$= 3[a_m 3^{2m} + a_{m-1} 3^{2m-1} + \dots + a_1 3^1]$

Thus $3 \mid n - s$ by definition S

Now suppose $3 \mid s$. Since $3 \mid n - s$ and $3 \mid s$, $3 \mid n - s + s$ by Thm 28. So $3 \mid n$.

Assume $n \geq 1$ is the small[est] $\mathbb{Z}$ w/ a p div

$9^2 = (3^2)^2 = 3^4$

$9^3 = (3^2)^3 = 3^6$

$3(3^2 + 3^3) = (3^2)^2 + (3^2)^3$

Attempt 4: After Second Class

29: $3|n$ if and only if $3|$ the ~~sum~~ sum of the digits of n. Suppose $n \in \mathbb{Z}$

Let $n = 10^0 a_0 + a_1 10 + a_2 100 + \dots + a_n 10^n$ where $a_0, a_1, a_2, \dots a_n$ represent the digits of n.

Suppose $3|n$, then ~~note~~

~~$10 \equiv_3 1$~~, $n \equiv_3 10^0 a_0 + 10 a_1 + 100 a^2 + \dots + 10^n a_n$

Since $10 \equiv_3$, $n \equiv_3 a_0 + a_1 + a_2 + \dots + a_n$

so $n \equiv_3 \sum_{i=1} a_i$ and $\sum_{i=1}^{n} a_i \equiv_3 0$

Thus 3 divides the sum of the digits of n.

Example

MAKING CONNECTIONS:	Beginning	Developing	Advancing
Between Definitions/Theorems	Recognizes some relevant definitions/theorems from the course or textbook with no attempts to connect them in their proving	Recognizes some relevant definitions/theorems from the course and attempts to connect them in their proving	Implements relevant definitions/theorems from the course and/or other resources outside the course in their proving
Between Representations[1]	Provides a representation with no attempts to connect it to another representation	Provides multiple representations and recognizes connections between representations	Provides multiple representations and uses connections between different representations
Between Examples	Generates one or two specific examples with no attempt to connect them	Generates one or two specific examples and recognizes a connection between them	Generates several specific examples and uses the key idea synthesized from their generation

Project 2: Implementing the CPR on Proving

- Setting
  - Transition-to-proof course in a Liberal Arts University in the West
  - 19 students (12 male, 7 female)
  - Taught using Inquiry-Based Learning pedagogy
  - Every student used a LiveScribe pen for her/his homework and notes
- Methods
  - Voluntary survey for all students after the course
  - Skype interviews with 4 students
  - Interviews fully transcribed

Students’ Feedback-1

- S1: I mean, **thinking about [a proof] in different ways** and proving it in different ways is the whole point of being a mathematician, is being able to prove something.
- S3: [The rubric] lets me know that, you know, it's okay to go between examples, it's **ok to do this, it's ok to do that.**
- S4: For example, if you have a proof, and you try a direct proof, well **try something different**! Do the contrapositive, or do the contradiction. You know, even if it may not work and in the end you spent an extra 20 or 30 minutes to do it, you know, it pays off in the end and it **builds your creativity**.

Students' Feedback-2

- S4: [W]ell, I would kind of use [the CPR on Proving] as a **checklist** to go through it and when I'm **evaluating my proof**, I would use and say "could I **make any connection**?" …You know, but could I do more? Could I do it better? Could I **go from developing to satisfactory** in my proof?
- S4: when **I got stuck** on the proof on a problem in the book, I would just look back to [the CPR on Proving], and 'oh let me try it this way, let me try it that way.

Fostering Creativity

- Non-judgmental environment
- Authority in creativity
- "To promote creativity and mathematical thinking, teachers should encourage good ideas even (and, in fact, especially) when a student suggests an unexpected answer or when the answers are inaccurate." (Hershkovitz, Peled, & Littler, 2009, p. 265)

Fostering Creativity (cont.)

- Problem Posing/Conjecturing
- Solutions as problems: “Students are rarely asked to view a solution to a problem as a starting point in problem solving” (Knuth, 2002, p. 129)

Creativity in Business

- The organizational climates that stimulate creativity (Amabile, 1988; Isaksen, 1995):
  - feel challenged by their goals, operations and tasks
  - feel able to take initiatives and to find relevant information
  - feel able to interact with others
  - feel that new ideas are met with support and encouragement
  - feel able to put forward new ideas and views
  - experience much debate within a prestige-free and open environment
  - feel uncertainty is tolerated and thus risk-taking is encouraged.

Recap of the Research

- We are investigating the relative, non-judgmental, process of creativity
- Two major actions that create potential for mathematical creativity:
  - Making Connections
  - Taking Risks
- Our conjecture is that, if both are fostered in any environment, creative (and eventually valid) products will be produced.

Future Research

- Implementation of CPR on Proving in the classroom
- Creativity x Neuroscience
  - EEG "reliving" with the proving process
- CPR on Problem Solving
- Creativity in a social setting
- CPR on Linear Algebra, Calculus, and…
  - …PRE-CALC?!?

It must not be forgotten that the basic law of children's creativity is that its value lies not in its results, not in the product of creation, but in the process itself. It is not important what children create, but that they do create, that they exercise and implement their creative imagination.

(Vygotsky, 2004, p. 72)

Thank you!

Questions?

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